HDG for Stokes: implementation

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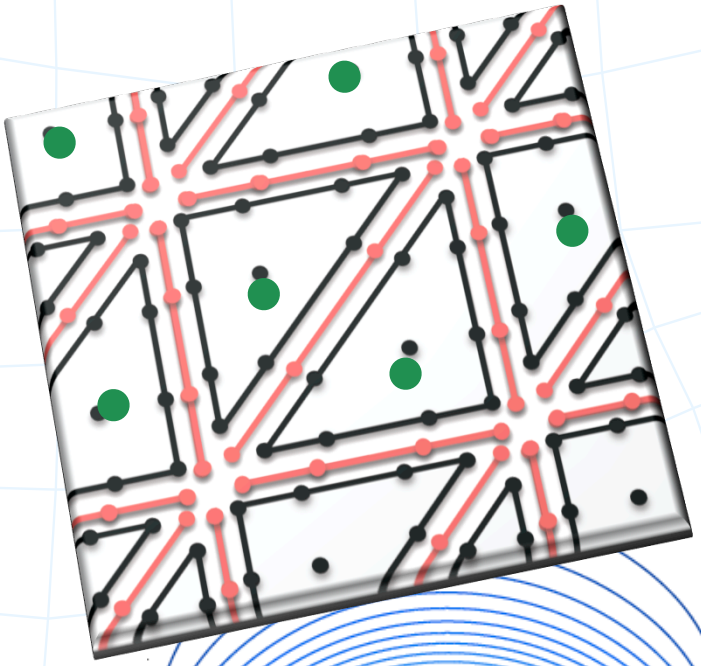
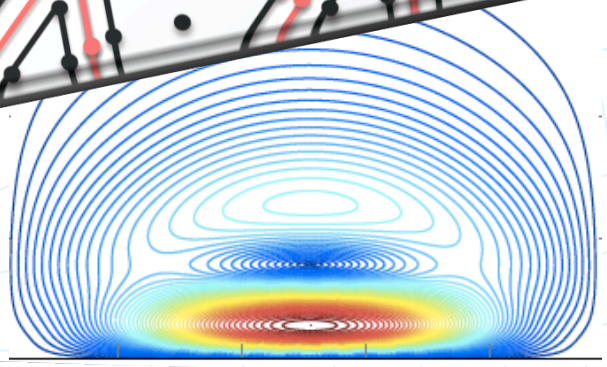
OUTLINE

1. Why HDG for incompressible flow?
   - Brief introduction
   - Comparison with CG

   (Results from the PhD thesis of Mahendra Paipuri IST-UPC)

2. HDG formulation for Stokes
   - Derivation of the weak form
   - Discretization, linear system
   - Matlab implementation



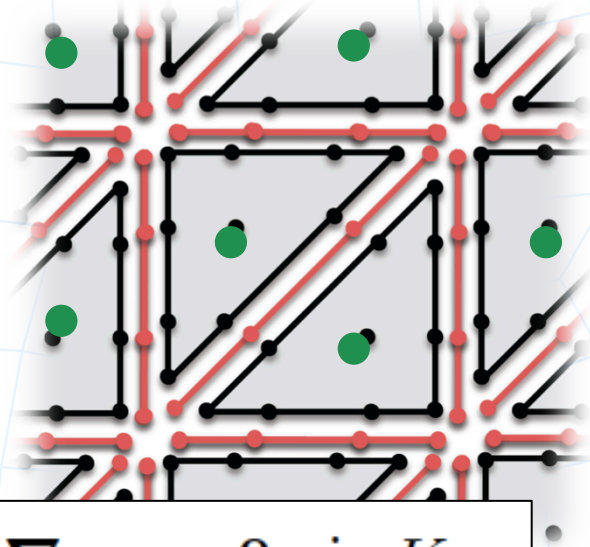


$$
\begin{aligned}
&(\delta \boldsymbol{L}, \boldsymbol{L}_h)_{\Omega_e} + (\operatorname{div} \delta \boldsymbol{L}, \boldsymbol{u}_h)_{\Omega_e} - \langle \delta \boldsymbol{L} \boldsymbol{n}, \hat{\boldsymbol{u}}_h \rangle_{\partial\Omega_e} = 0, \\
&-(\operatorname{grad} \delta \boldsymbol{u}, \boldsymbol{u}_h \otimes \boldsymbol{u}_h)_{\Omega_e} + (\delta \boldsymbol{u}, \operatorname{div} (-\nu \boldsymbol{L}_h + p_h \boldsymbol{I}))_{\Omega_e} \\
&\qquad + \langle \delta \boldsymbol{u}, (\hat{\boldsymbol{u}}_h \otimes \hat{\boldsymbol{u}}_h) \boldsymbol{n} + \tau (\boldsymbol{u}_h - \hat{\boldsymbol{u}}_h) \rangle_{\partial\Omega_e} - (\delta \boldsymbol{u}, \boldsymbol{f})_{\Omega_e} = 0 \\
&-(\operatorname{grad} \delta p, \boldsymbol{u}_h)_{\Omega_e} + \langle \delta p, \hat{\boldsymbol{u}}_h \cdot \boldsymbol{n} \rangle_{\partial\Omega_e} = 0, \\
&\frac{1}{|\partial\Omega_e|} \langle p_h, 1 \rangle_{\partial\Omega_e} = \rho_e,
\end{aligned}
$$

HDG for INCOMPRESSIBLE FLOW

[Cockburn, Nguyen & Peraire, JSC 2010]

$$
\begin{aligned}
-\nabla \cdot (\nu \nabla \boldsymbol{u}) + \nabla p &= \boldsymbol{f} && \text{in } \Omega \\
\nabla \cdot \boldsymbol{u} &= 0 && \text{in } \Omega \\
\boldsymbol{u} &= \boldsymbol{u}_D && \text{on } \partial\Omega
\end{aligned}
$$



- Local problems:

$$
\boldsymbol{L} - \nabla \boldsymbol{u} = 0, \quad \nabla \cdot (-\nu \boldsymbol{L} + p\mathbf{I}) = \boldsymbol{f} \quad \text{and} \quad \nabla \cdot \boldsymbol{u} = 0 \quad \text{in } K_i
$$

$$
\boldsymbol{u} = \hat{\boldsymbol{u}} \quad \text{on } \partial K_i
$$

Dirichlet problem in each element K_i with data $\hat{\mathbf{u}}$ and ρ_i

$$
\frac{1}{|K_i|} \int_{K_i} p \, dV = \rho_i
$$

- Global equations: conservativity, well-posedness and BC

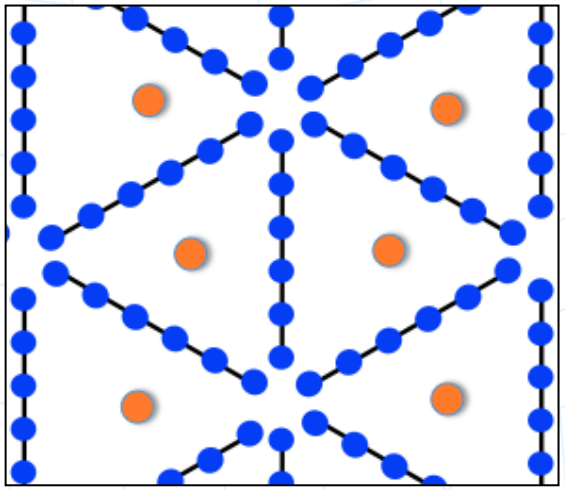
$$
[\![(-\nu \boldsymbol{L} + p\mathbf{I}) \cdot \boldsymbol{n}]\!] = 0 \quad \text{on } \Gamma \backslash \partial\Omega
$$

$$
\int_{\partial K_i} \hat{\boldsymbol{u}} \cdot \boldsymbol{n} \, dS = 0 \quad \text{for } i = 1, \ldots, \mathrm{n_{el}}
$$

$$
\hat{\boldsymbol{u}} = \boldsymbol{u}_D \quad \text{on } \partial\Omega \quad \text{and} \quad \sum_{i=1}^{\mathrm{n_{el}}} |K_i| \rho_i = |\Omega| \rho_\Omega
$$

HDG for incompressible flow

$$-\nabla \cdot (\nu \nabla \boldsymbol{u}) + \nabla p = \boldsymbol{f}$$
$$\nabla \cdot \boldsymbol{u} = 0$$

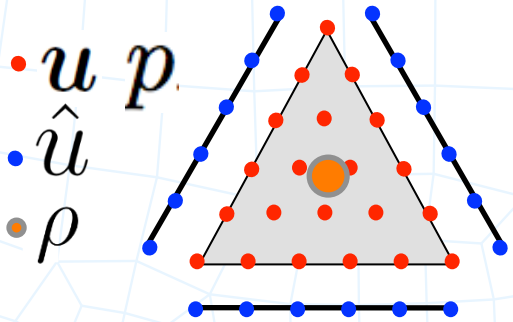


1. Global problem: involves only
   - $\hat{u}$: velocity trace
   - ρ: mean of the pressure in each element

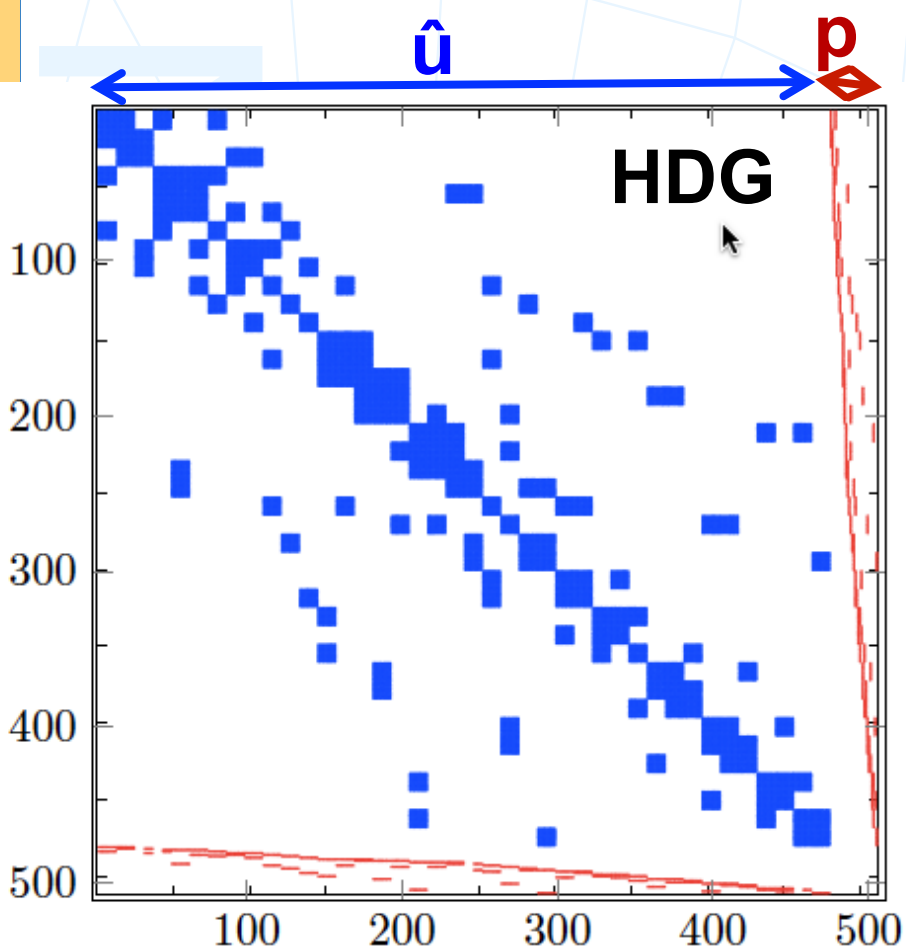
$$\rho^K = \frac{1}{\ell(\partial K)} \int_{\partial K} p \, d\Omega$$

2. Element-by-element postprocess (local problem)

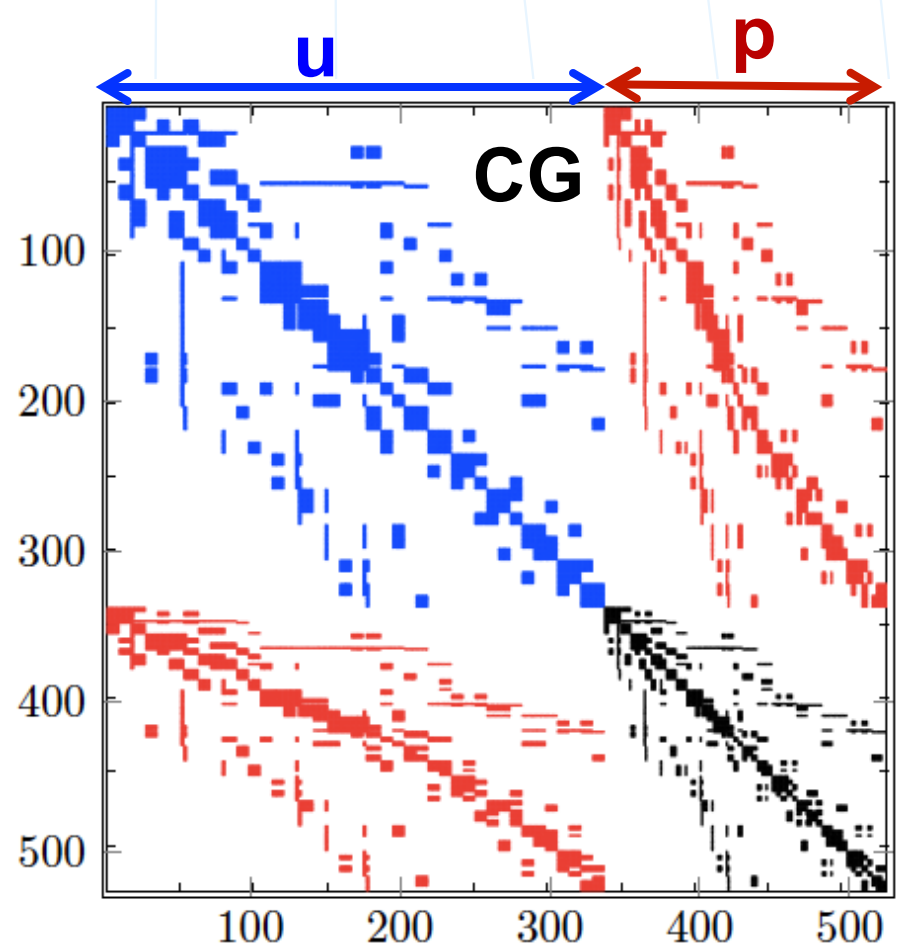
$$\hat{\boldsymbol{u}} \; \rho^K \rightarrow \boldsymbol{u}^K \; p^K$$



Sparsity pattern HDG (P_k) and CG (P_kP_{k-1})
for a regular mesh with degree k=5



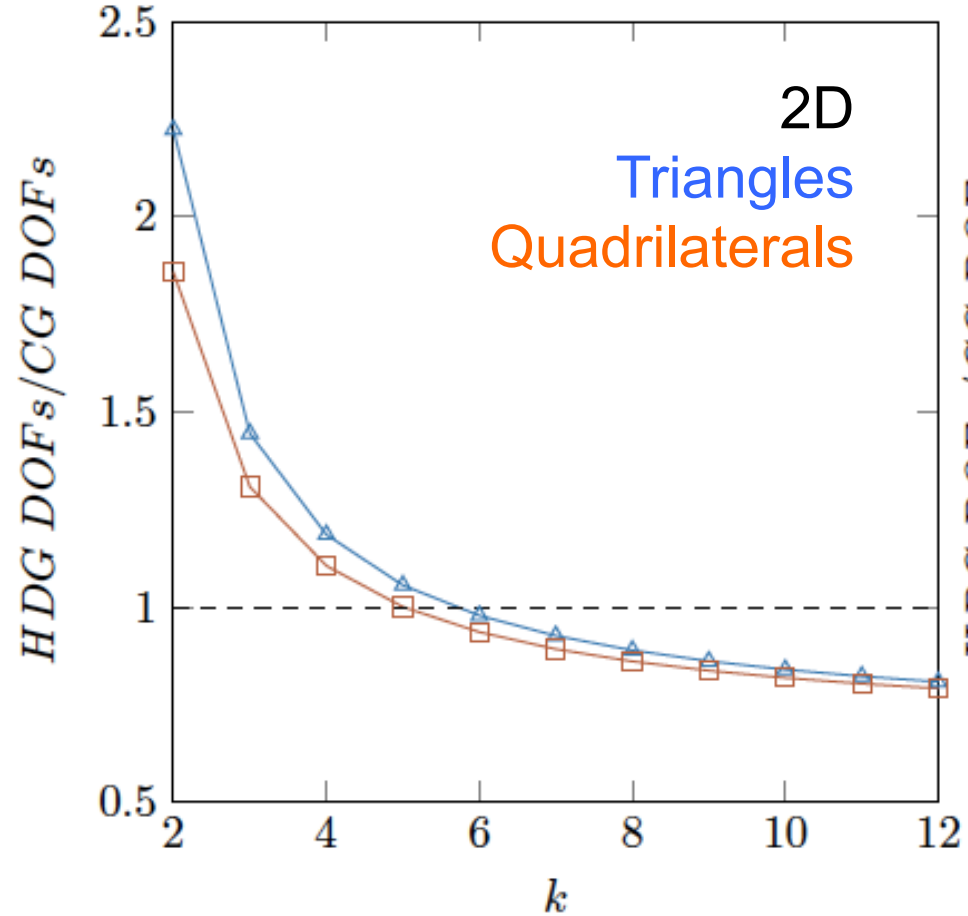
(a) Sparsity pattern of HDG, DOFs = 511, nnz = 1830.

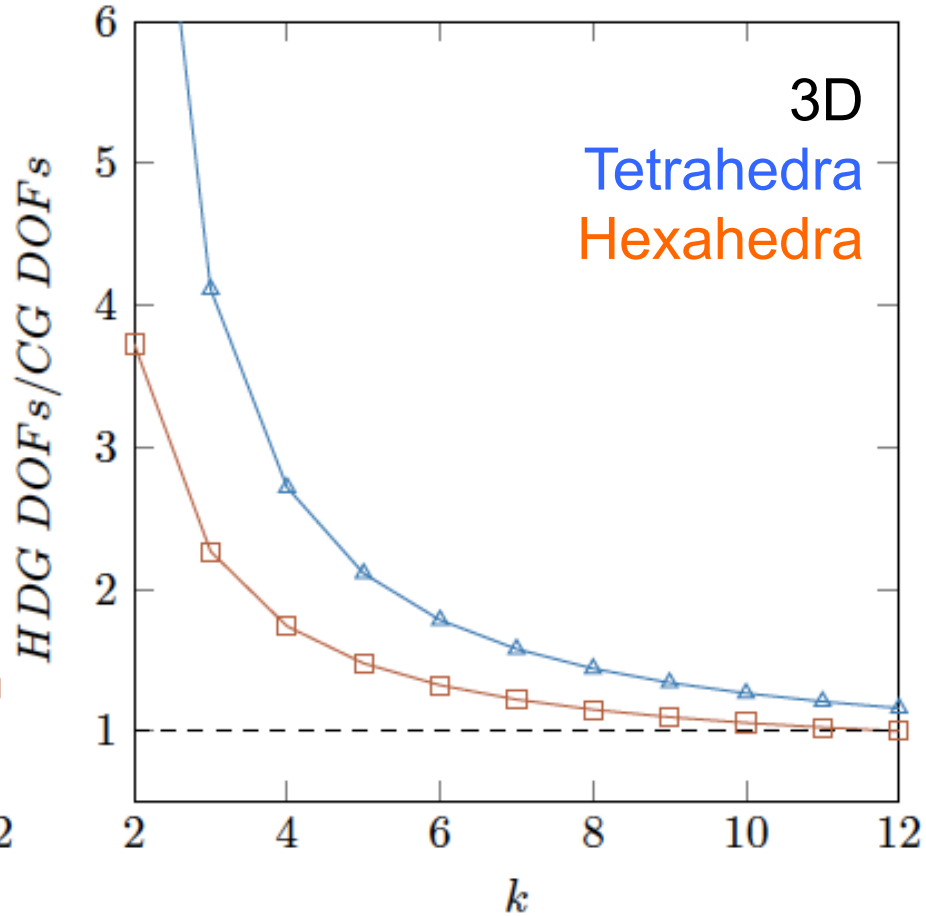


(b) Sparsity pattern of CG, DOFs = 530, nnz = 3580.

Count of DOFs

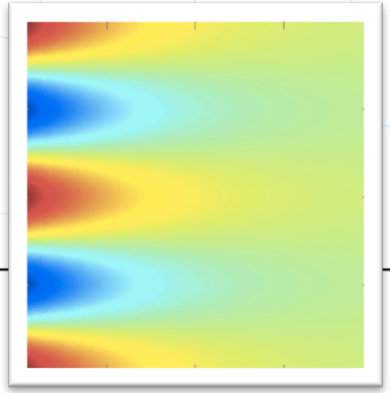
- Same hypotheses as [Huerta, Angeloski, Roca and Peraire, IJNME 2013] for number of geometrical entities in terms of number of elements
- HDG with degree k, CG with Taylor-Hood k-(k-1)

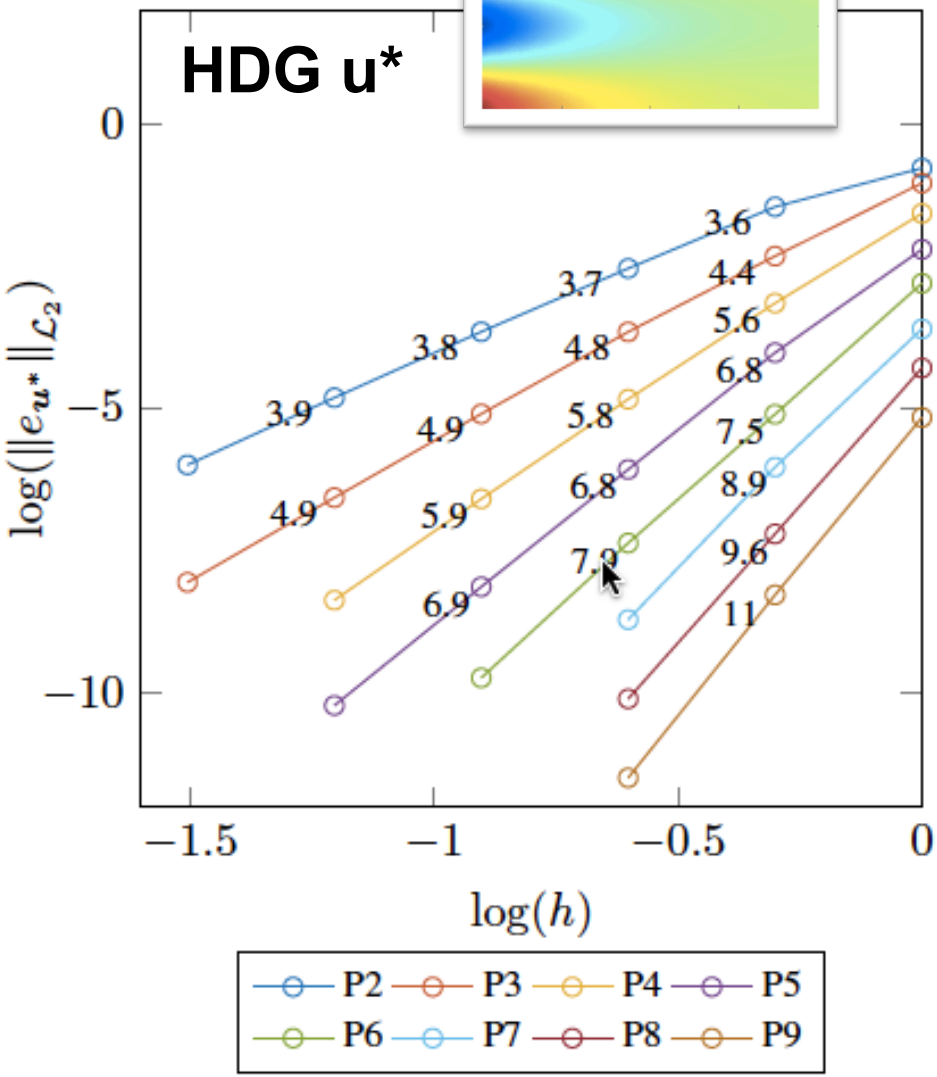


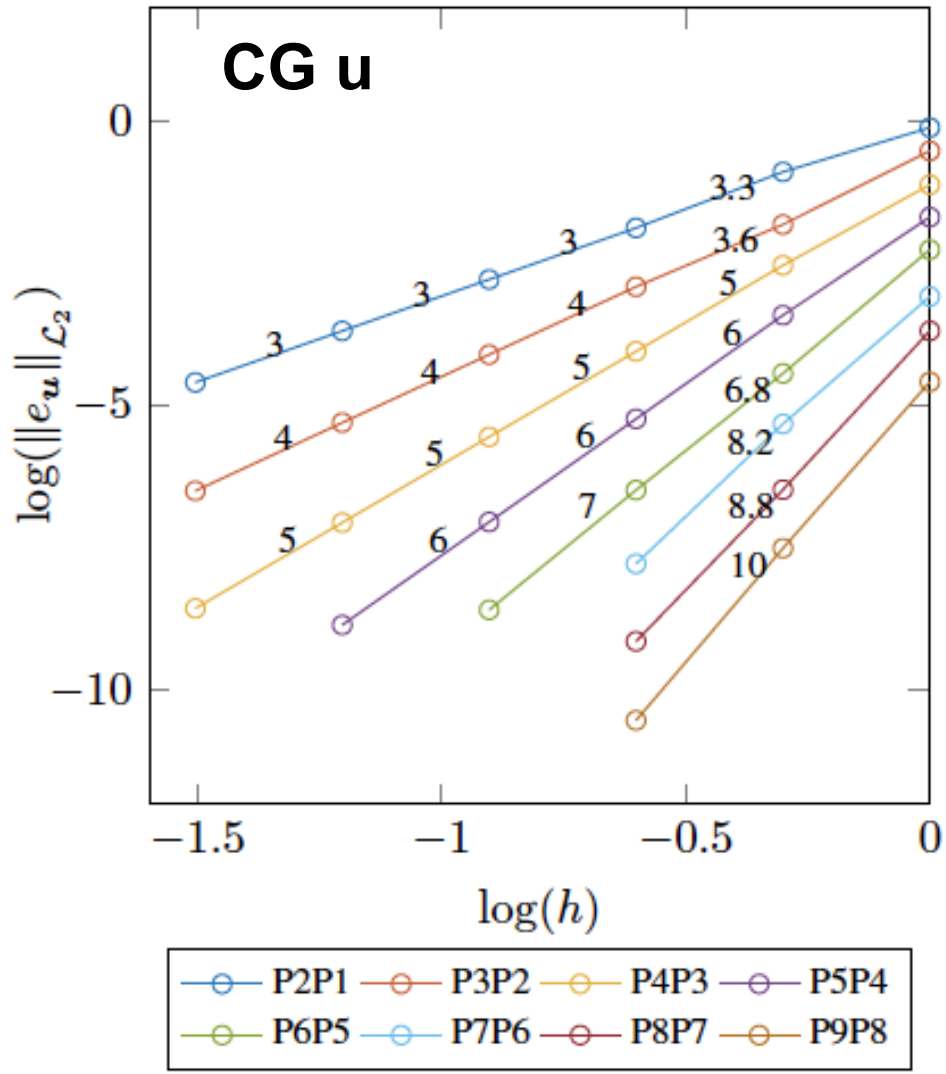


But… different convergence rates… :-/

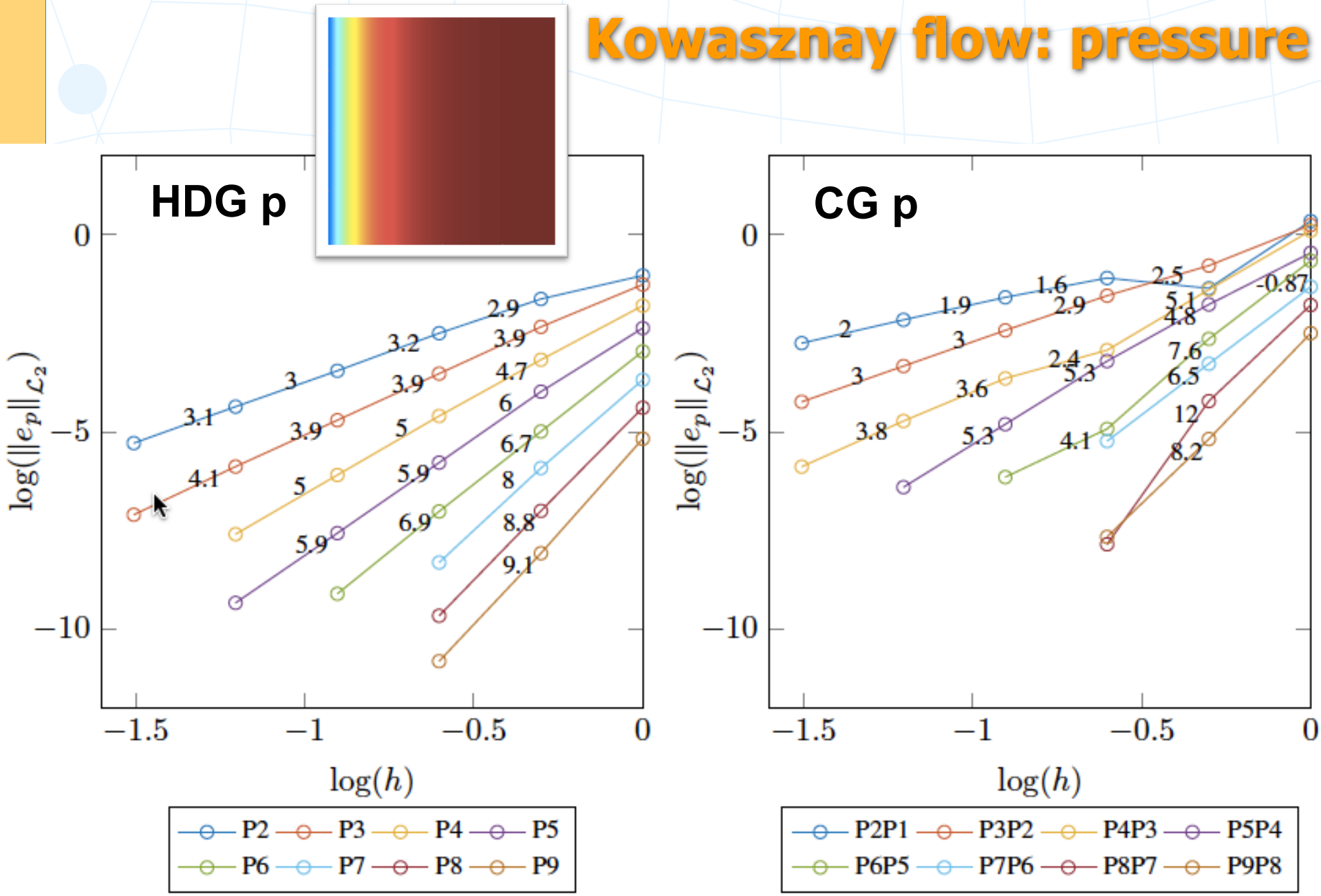
Kowasznay flow: velocity





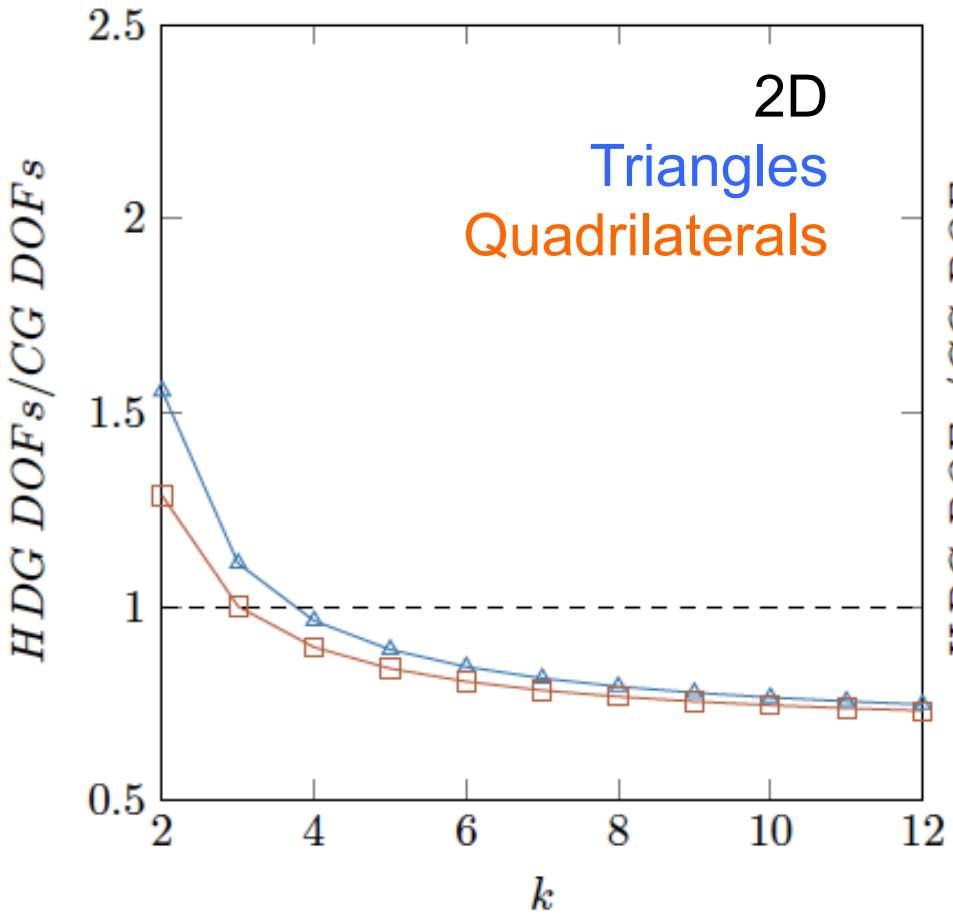


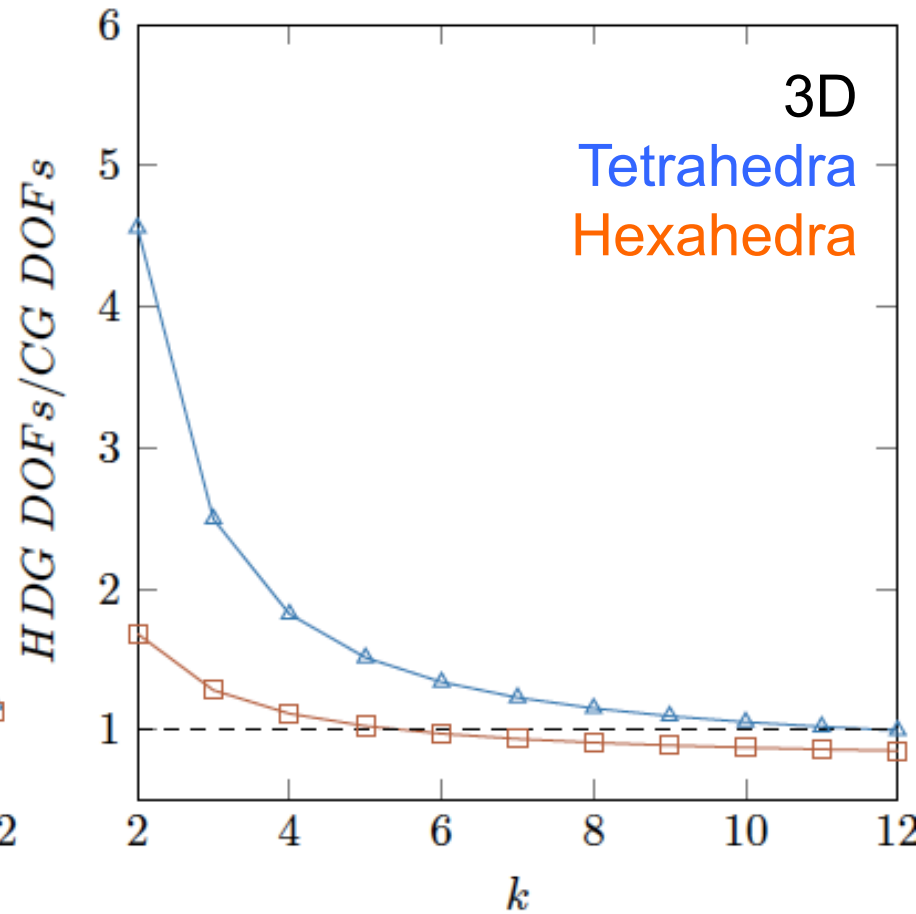
Kowasznay flow: pressure



Count of DOFs for same converge rate

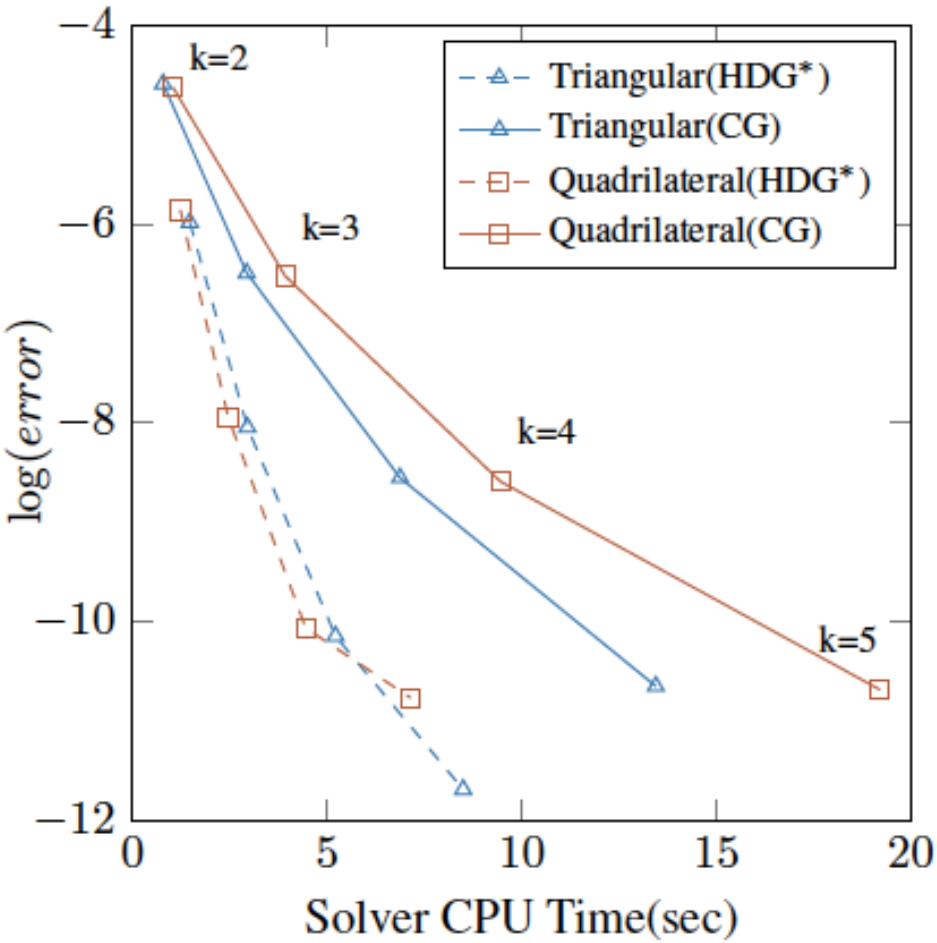
- HDG with degree **k-1**, CG with Taylor-Hood **k-(k-1)**



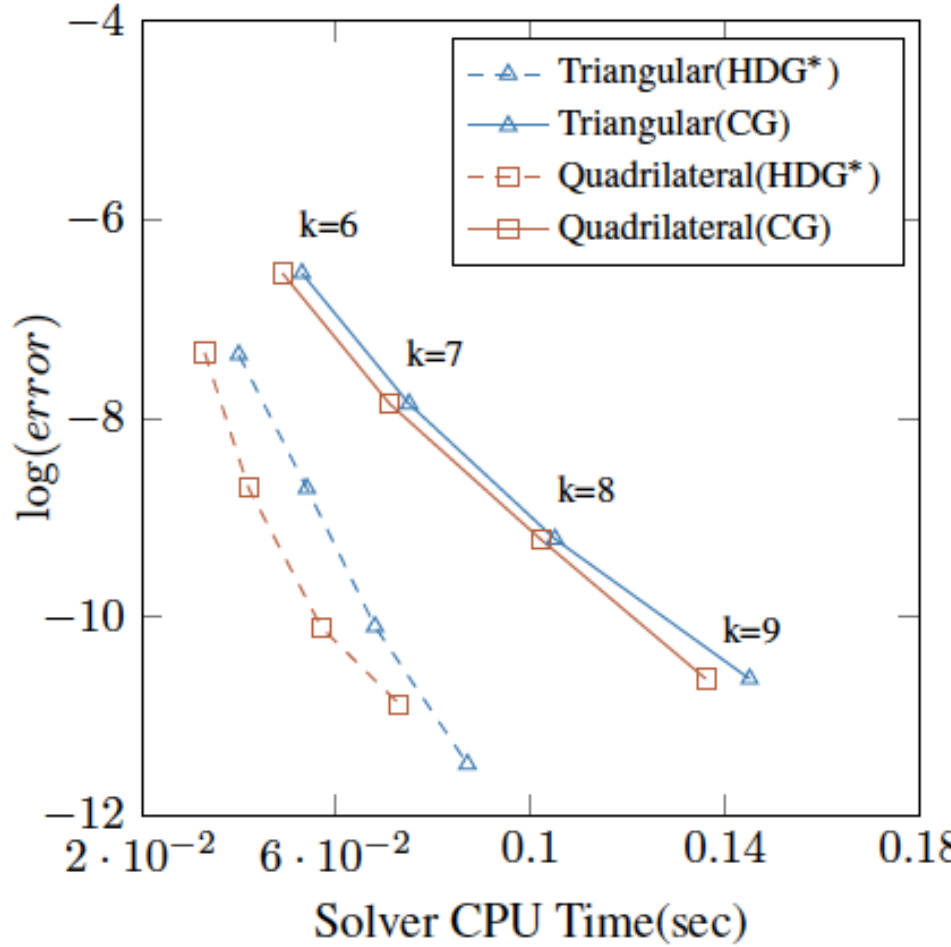


- But… which one is more efficient in terms of CPU time and accuracy?

Kowasznay flow: CPU time for linear solver v.s. velocity error

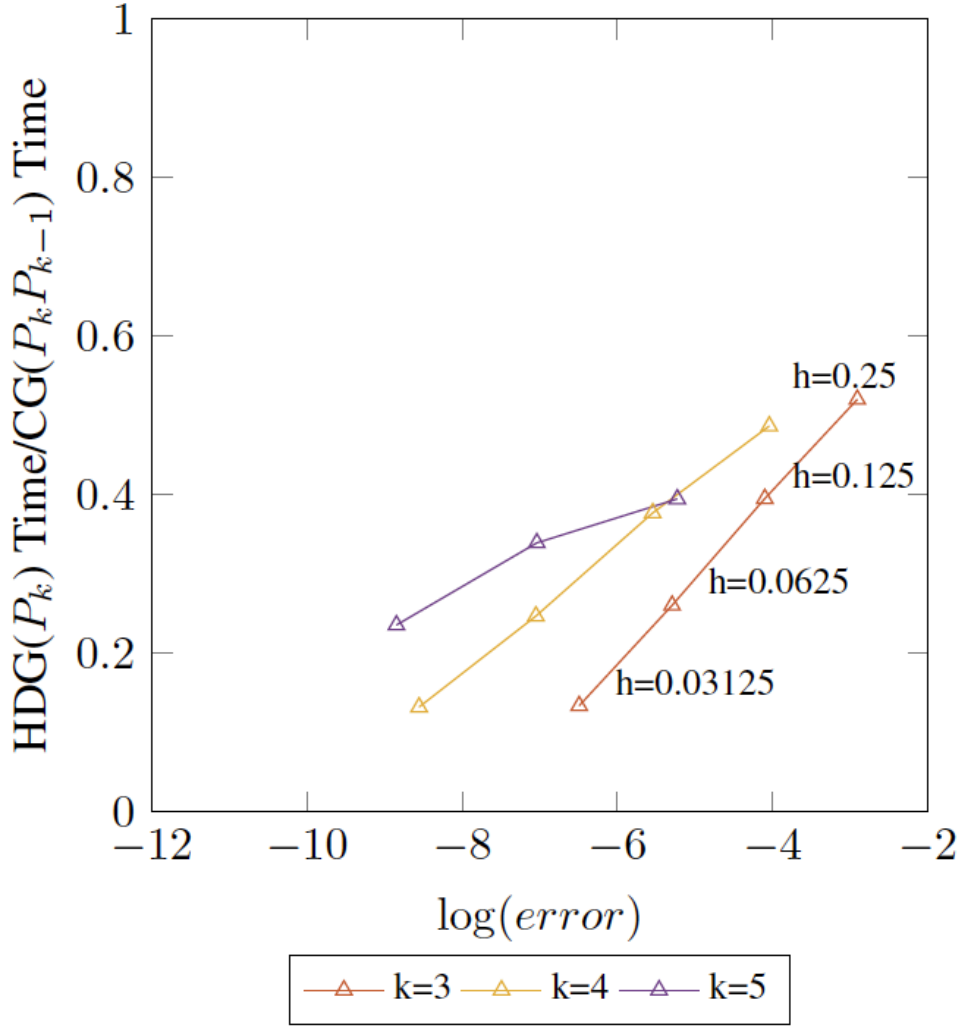


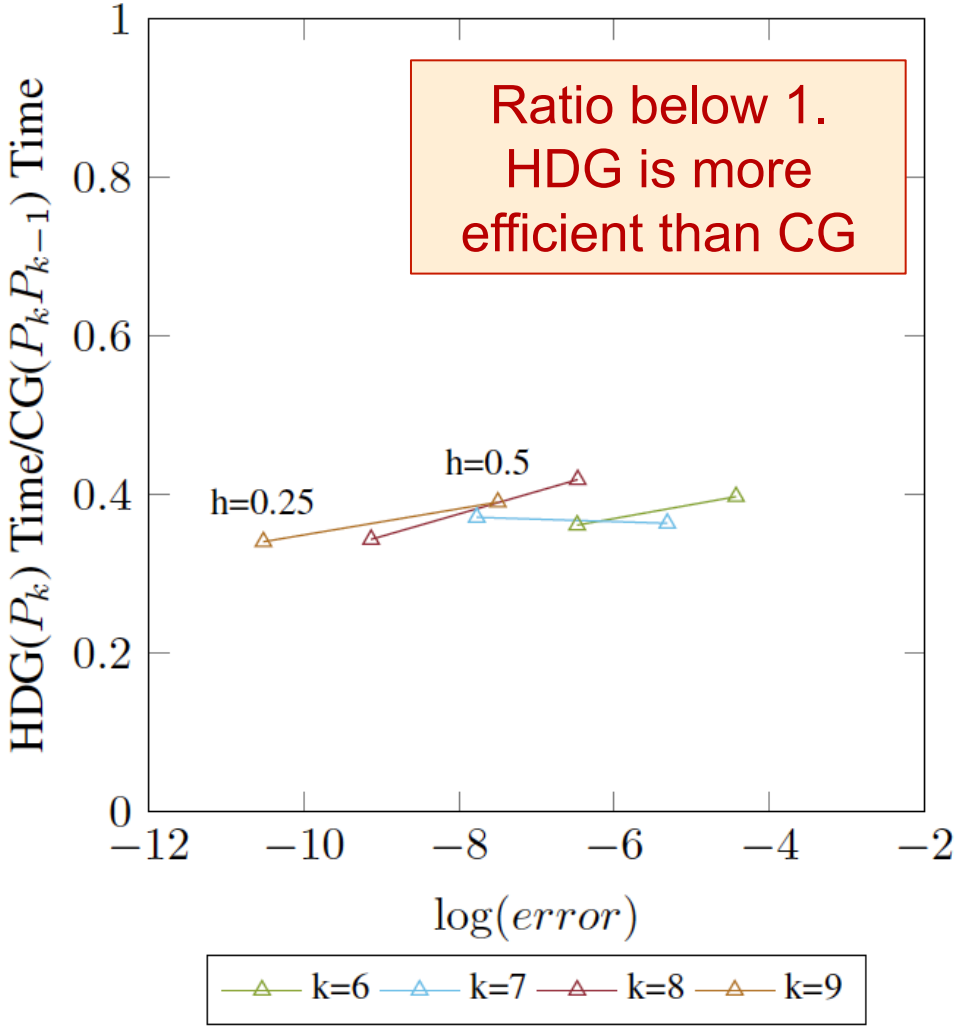
(a) $h = 0.03125$.



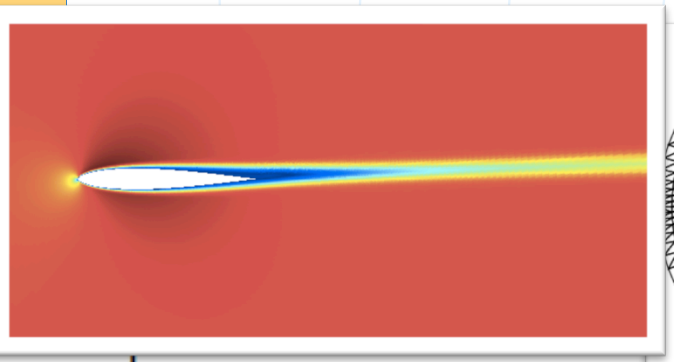
(b) $h = 0.25$.

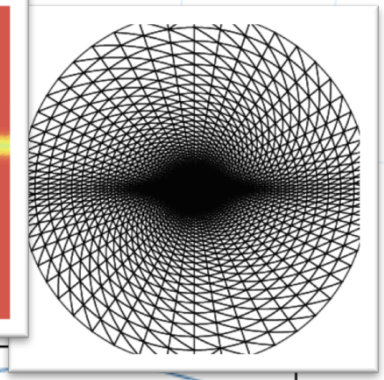
Kowasznay flow: ratio of CPU time for linear solver v.s. velocity error

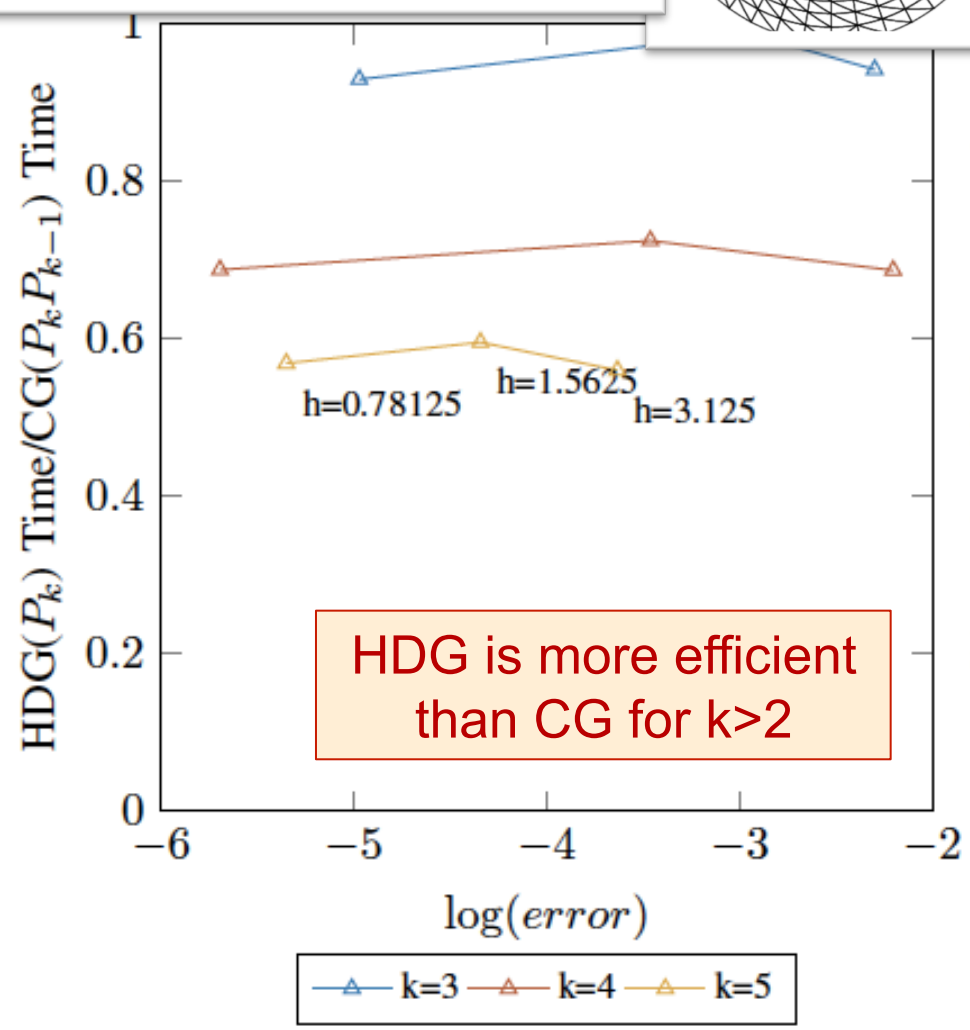




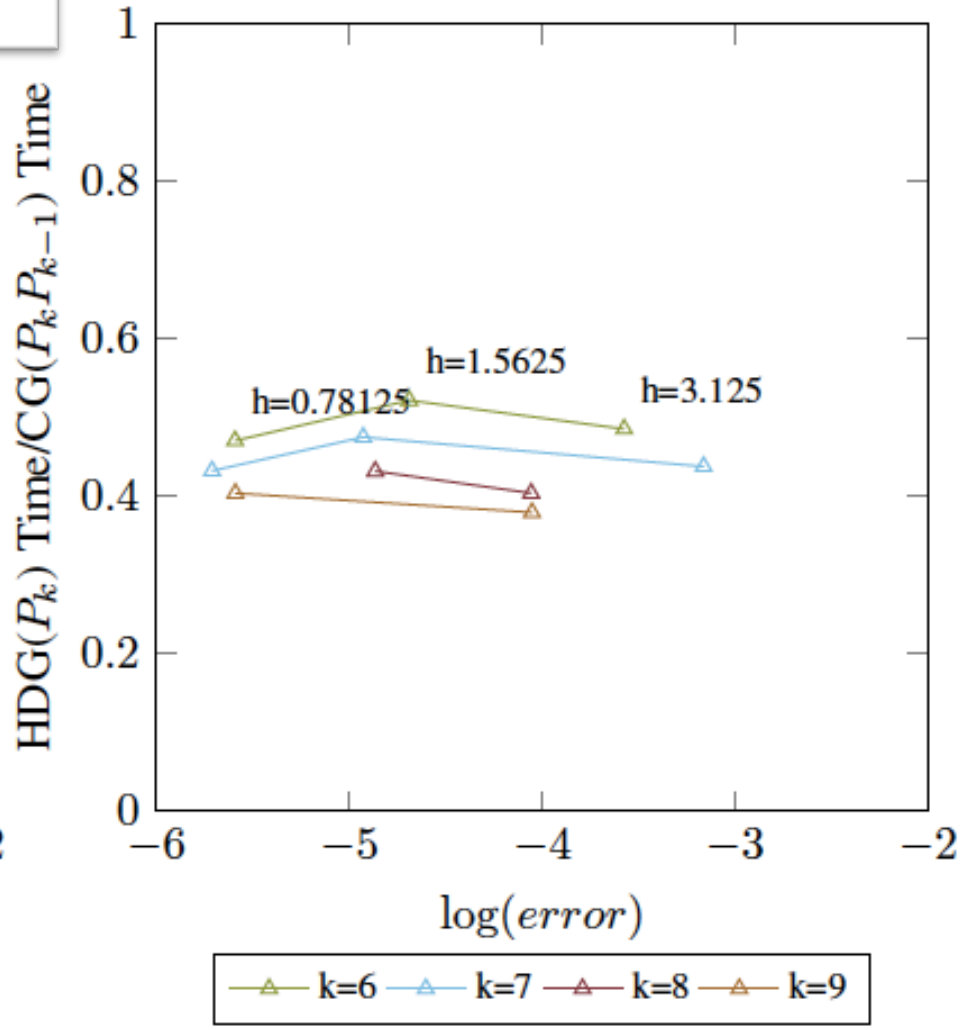
NACA 0012: ratio of CPU time for linear solver v.s. error in C_L





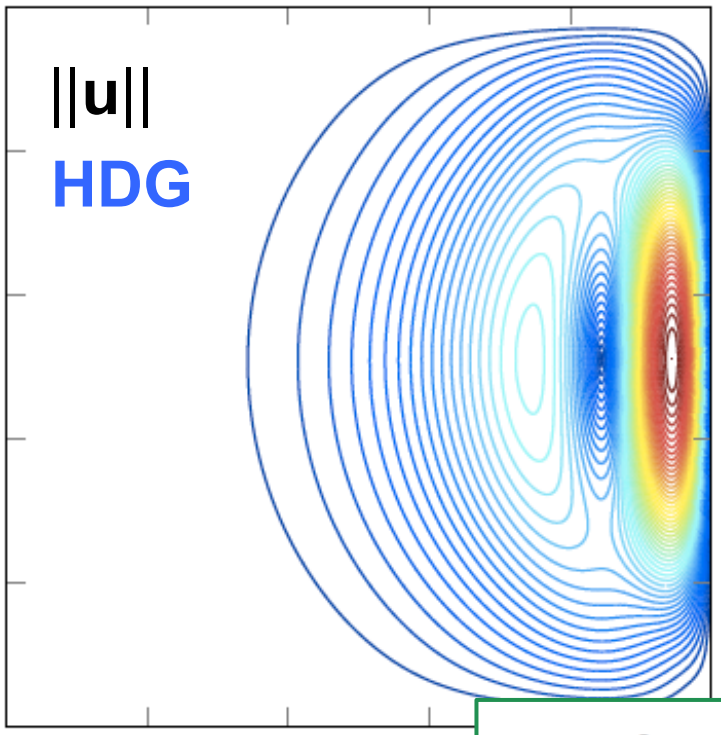


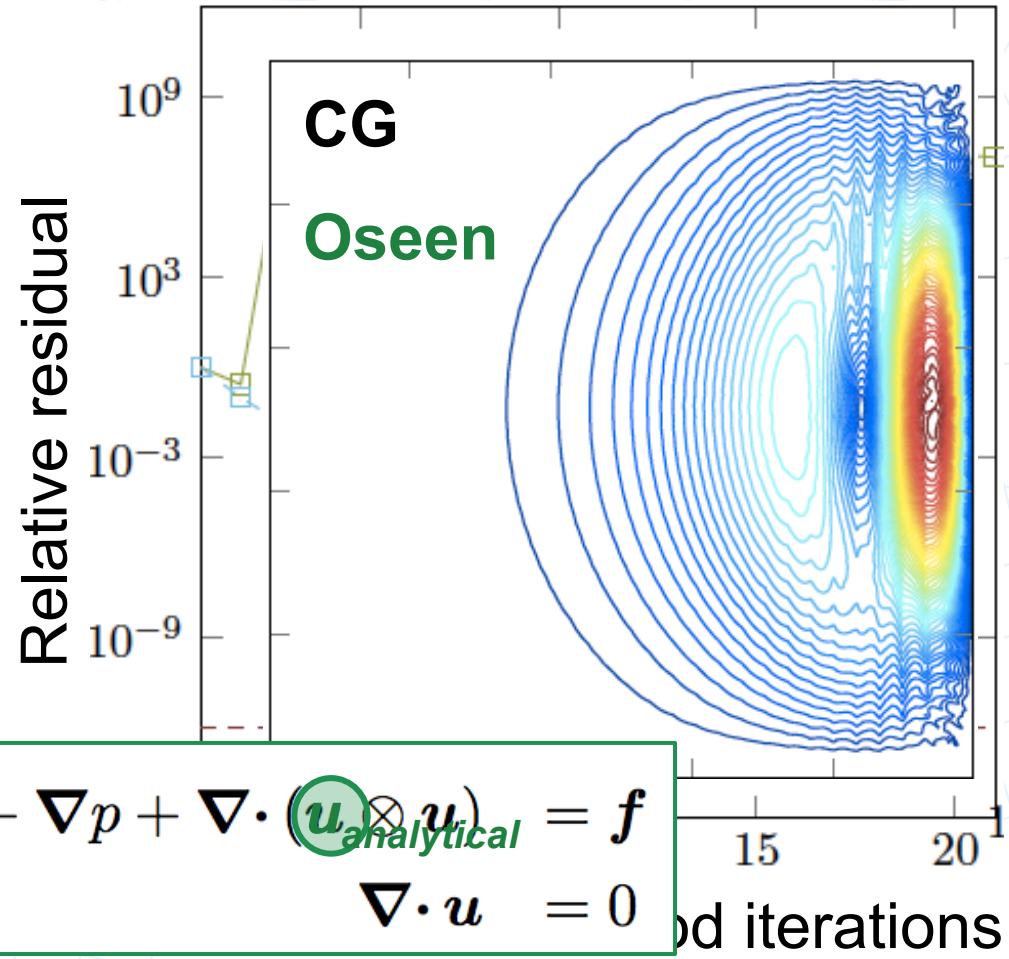
HDG is more efficient than CG for k>2



STABILITY in the presence of sharp fronts

- HDG solution of the Navier-Stokes equations (Re=2000, k=3)



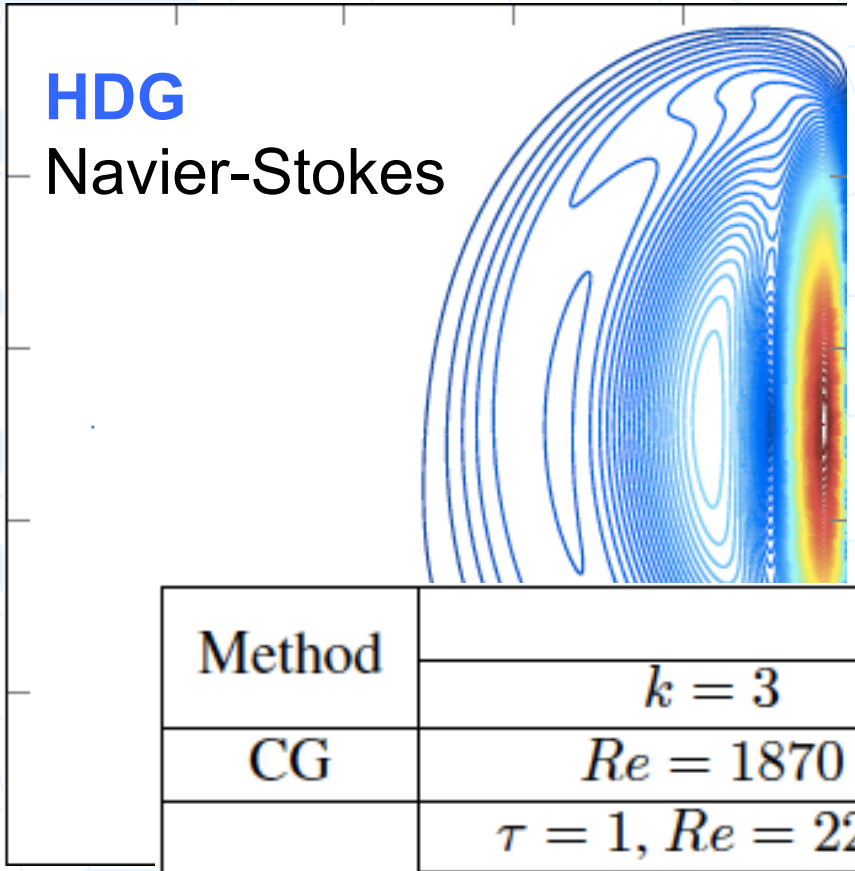


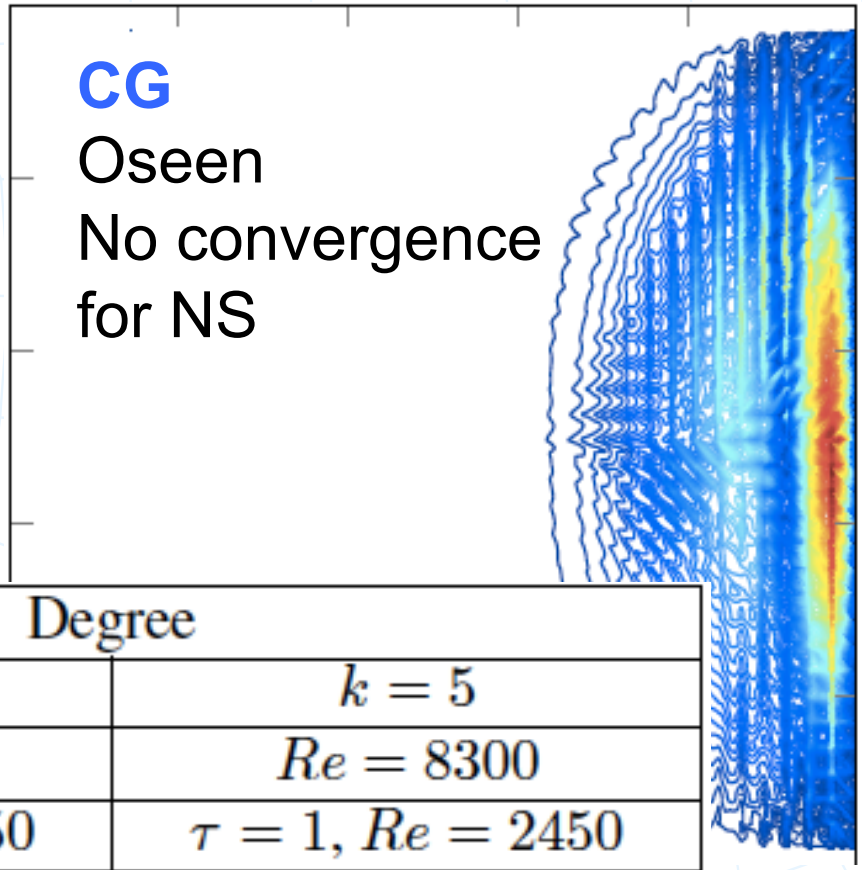
$$-\nu\nabla^2\boldsymbol{u} + \boldsymbol{\nabla}p + \boldsymbol{\nabla}\cdot(\boldsymbol{u}\otimes\boldsymbol{u}) = \boldsymbol{f}$$
$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$$

$\boldsymbol{u}_{analytical}$

STABILITY in the presence of sharp fronts

- HDG solution, Navier-Stokes equations (Re=11100, k=3)



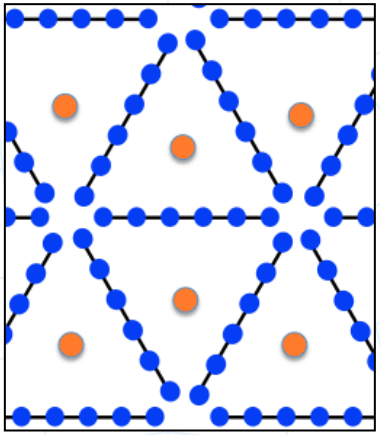


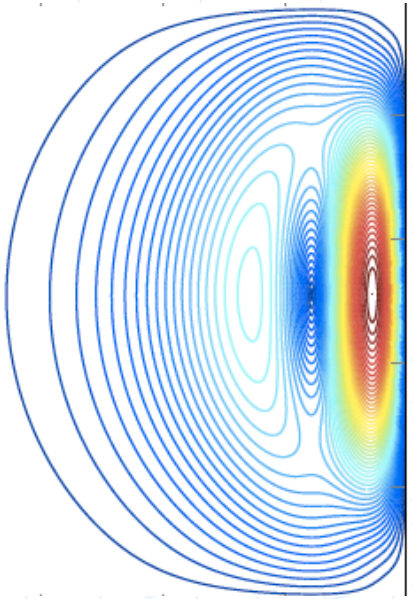
Method	Degree	
	$k = 3$	$k = 5$
CG	$Re = 1870$	$Re = 8300$
HDG	$\tau = 1$, $Re = 2250$	$\tau = 1$, $Re = 2450$
	$\tau = 100$, $Re = 8000$	$\tau = 100$, $Re = 9100$
	$\tau = 500$, $Re = 11100$	$\tau = 500$, $Re = 15400$

Conclusions HDG vs CG

- HDG has similar (or better) computational efficiency compared to CG, outperforming other DG methods:
  - #DOF slightly larger than CG for Laplace
  - more DOF for velocity and less for pressure for incompressible flow

  and
  - superconvergence, better accuracy
  - nice block structure of matrices, convenient for direct solvers





- HDG inherits DG stability properties in the presence of sharp fronts.

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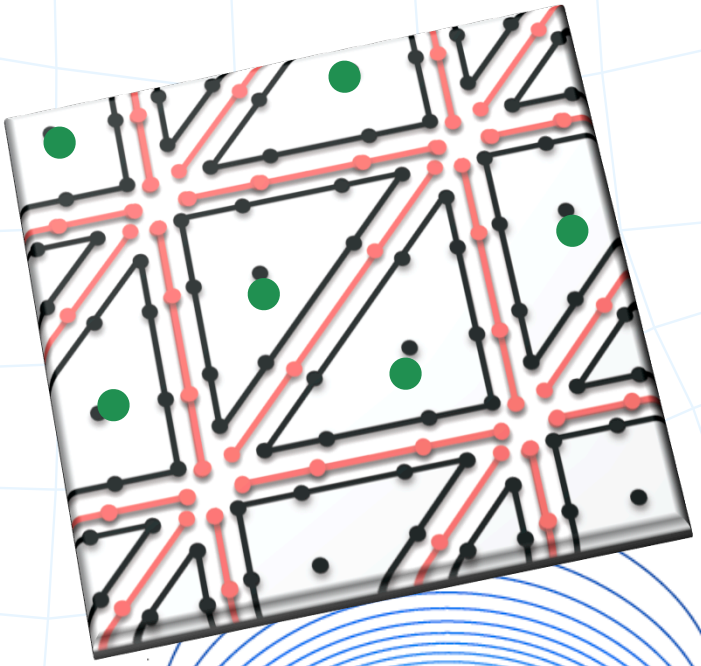
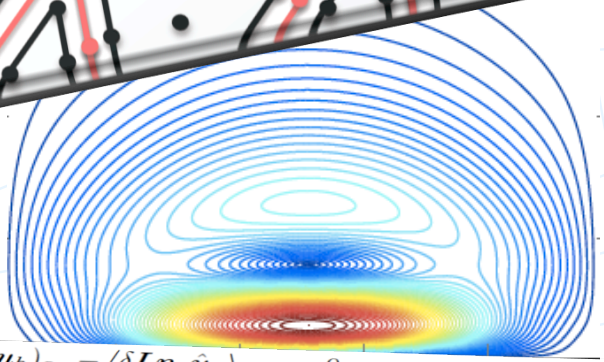


2. **HDG formulation for Stokes**
   - Derivation of the weak form
   - Discretization, linear system
   - Matlab implementation



$$
\begin{aligned}
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&\qquad + \langle \delta \boldsymbol{u}, (\hat{\boldsymbol{u}}_h \otimes \hat{\boldsymbol{u}}_h) \boldsymbol{n} + \tau (\boldsymbol{u}_h - \hat{\boldsymbol{u}}_h) \rangle_{\partial\Omega_e} - (\delta \boldsymbol{u}, \boldsymbol{f})_{\Omega_e} = 0 \\
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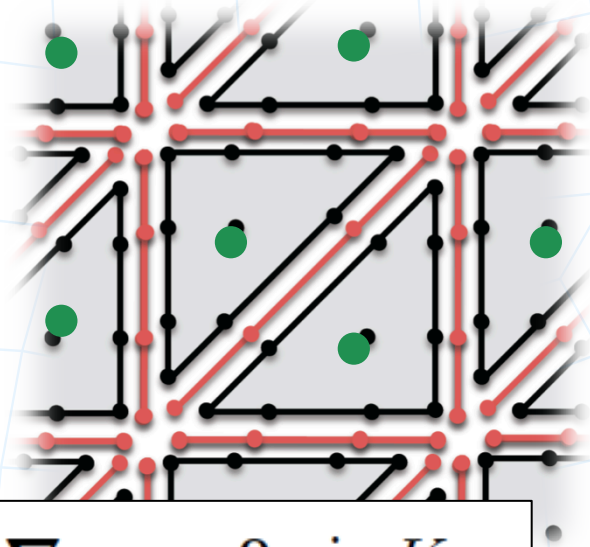

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Dirichlet problem in each element K_i with data $\hat{\mathbf{u}}$ and ρ_i

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$$
\hat{\boldsymbol{u}} = \boldsymbol{u}_D \quad \text{on } \partial\Omega \quad \text{and} \quad \sum_{i=1}^{\mathrm{n_{el}}} |K_i| \rho_i = |\Omega| \rho_\Omega
$$

Local problem: weak form

- Dirichlet problem in each element

$$\boldsymbol{L} - \boldsymbol{\nabla}\boldsymbol{u} = \boldsymbol{0}, \quad \boldsymbol{\nabla}\cdot(-\nu\boldsymbol{L} + p\mathbf{I}) = \boldsymbol{f} \;\text{ and }\; \boldsymbol{\nabla}\cdot\boldsymbol{u} = \boldsymbol{0} \quad \text{in } K_i$$

$$\boldsymbol{u} = \hat{\boldsymbol{u}} \quad \text{on } \partial K_i$$

$$\frac{1}{|K_i|}\int_{K_i} p\,dV = \rho_i$$

- Weak form: given $\hat{u}$ and ρ_i, find u, L and p such that

$$\int_{K_i} (-\boldsymbol{\nabla}\cdot(\nu\boldsymbol{L}_h) + \boldsymbol{\nabla}p_h)\cdot\boldsymbol{v}\,dV + \int_{\partial K_i} \tau\nu(\boldsymbol{u}_h - \hat{\boldsymbol{u}}_h)\cdot\boldsymbol{v}\,dS = \int_{K_i} \boldsymbol{f}\cdot\boldsymbol{v}\,dV$$

$$\int_{K_i} \boldsymbol{L}_h : \boldsymbol{Q}\,dV + \int_{K_i} (\boldsymbol{\nabla}\cdot\boldsymbol{Q})\cdot\boldsymbol{u}_h\,dV - \int_{\partial K_i} (\boldsymbol{Q}\cdot\boldsymbol{n})\cdot\hat{\boldsymbol{u}}_h\,dS = 0$$

$$\int_{K_i} \boldsymbol{u}_h\cdot\boldsymbol{\nabla}q\,dV - \int_{\partial K_i} (\hat{\boldsymbol{u}}_h\cdot\boldsymbol{n})q\,dS = 0$$

$$\frac{1}{|K_i|}\int_{\partial K_i} p_h\,dV = \rho_i, \qquad \boxed{\widehat{\boldsymbol{L}} := \boldsymbol{L} + \tau(\hat{\boldsymbol{u}} - \boldsymbol{u})\otimes\boldsymbol{n}}$$

for all v, Q and q

Local problem: discretization

- Linear system in each element with a Lagrange multiplier λ

$$\begin{bmatrix} \mathbf{A}^i_{uu} & \mathbf{A}^i_{uL} & \mathbf{A}^i_{up} & \mathbf{0} \\ \mathbf{A}^i_{Lu} & \mathbf{A}^i_{LL} & \mathbf{0} & \mathbf{0} \\ \mathbf{A}^i_{pu} & \mathbf{0} & \mathbf{0} & \mathbf{A}^i_{\rho p}{}^T \\ \mathbf{0} & \mathbf{0} & \mathbf{A}^i_{\rho p} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u}^i \\ \mathbf{L}^i \\ \mathbf{p}^i \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{f}^i_u \\ \mathbf{f}^i_L \\ \mathbf{f}^i_p \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ 1 \end{bmatrix} \rho_i - \begin{bmatrix} \mathbf{A}^i_{u\widehat{u}} \\ \mathbf{A}^i_{L\widehat{u}} \\ \mathbf{A}^i_{p\widehat{u}} \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} \widehat{\mathbf{u}}^{\mathbf{F}_{i1}} \\ \vdots \\ \widehat{\mathbf{u}}^{\mathbf{F}_{in}} \end{bmatrix}$$

- Local solver

$$\begin{bmatrix} \mathbf{u}^i \\ \mathbf{L}^i \\ \mathbf{p}^i \\ \lambda \end{bmatrix} = [\text{localSolverMat}] \begin{bmatrix} \widehat{\mathbf{u}}^{\mathbf{F}_{i1}} \\ \vdots \\ \widehat{\mathbf{u}}^{\mathbf{F}_{in}} \\ \rho_i \end{bmatrix} + [\text{localSolverVec}]$$

Global problem: weak form

#1 $$\boxed{[\![(-\nu \boldsymbol{L} + p\mathbf{I}) \cdot \boldsymbol{n}]\!] = 0 \quad \text{on } \Gamma \backslash \partial\Omega}$$

#2 $$\int_{\partial K_i} \hat{\boldsymbol{u}} \cdot \boldsymbol{n}\, dS = 0 \quad \text{for } i = 1, \ldots, \mathtt{n_{el}}$$

#3 $$\hat{\boldsymbol{u}} = \boldsymbol{u}_D \ \text{ on } \partial\Omega \ \text{ and } \ \sum_{i=1}^{\mathtt{n_{el}}} |K_i| \rho_i = |\Omega| \rho_\Omega$$

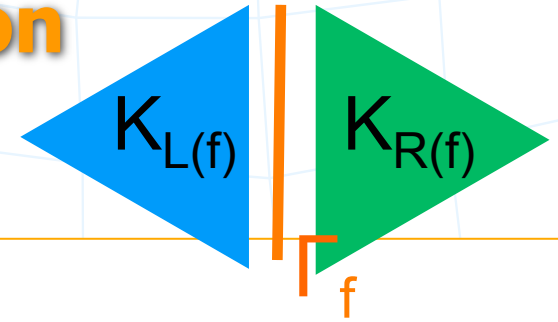
- Weak form equation #1

$$[\![\circledcirc]\!] = \circledcirc_{L(f)} + \circledcirc_{R(f)}$$
$$\{\circledcirc\} = \frac{1}{2}\left(\circledcirc_{L(f)} + \circledcirc_{R(f)}\right)$$

$$\int_\Gamma \widehat{\boldsymbol{v}} \cdot [\![(-\nu \boldsymbol{L}_h + p_h \mathbf{I}) \cdot \boldsymbol{n}]\!]\, dS + 2\int_\Gamma \widehat{\boldsymbol{v}} \cdot \left(\{\nu\tau \boldsymbol{u}_h\} - \{\nu\tau\}\hat{\boldsymbol{u}}_h\right) dS = 0$$

or equivalent expression as sum in elements…

Global problem: discretization



- Equation #1 on each face f

$$\mathbf{A}^{f,L}_{\widehat{u}u}\mathbf{u}^{L(f)} + \mathbf{A}^{f,L}_{\widehat{u}\boldsymbol{L}}\boldsymbol{L}^{L(f)} + \mathbf{A}^{f,L}_{\widehat{u}p}\mathbf{p}^{L(f)} + \mathbf{A}^{f,R}_{\widehat{u}u}\mathbf{u}^{R(f)} + \mathbf{A}^{f,R}_{\widehat{u}\boldsymbol{L}}\boldsymbol{L}^{R(f)} + \mathbf{A}^{f,R}_{\widehat{u}p}\mathbf{p}^{R(f)} + \mathbf{A}^{f}_{\widehat{u}\widehat{u}}\widehat{\mathbf{u}}^{f} = 0$$

replacing the local solver… leads to an equation with only $\hat{u}^i$ for the 5 faces, and the mean of the pressure in the two elements, $\rho^{L(f)}$ and $\rho^{R(f)}$.

- Equation #2 on each element i

$$\int_{\partial K_i} \hat{\boldsymbol{u}}_h \cdot \boldsymbol{n}\, dS = 0 \quad \text{for } i = 1, \ldots, \mathrm{n_{el}}$$

$$\mathbf{A}_{\rho\widehat{u}} \begin{bmatrix} \widehat{\mathbf{u}}^{\mathbf{F}_{i1}} \\ \vdots \\ \widehat{\mathbf{u}}^{\mathbf{F}_{in}} \end{bmatrix} = 0 \quad \text{for } i = 1, \ldots, \mathrm{n_{el}}$$

Implementation (similar to Laplace)

- Loop in elements K_i
  - Computation of elemental matrices (volume integrals and face integrals): A_{uu}, A_{lu},…
  - Computation of local solver: [localSolverMat] and [localSolverVec] such that

$$\begin{bmatrix} \mathbf{u}^i \\ \mathbf{L}^i \\ \mathbf{p}^i \\ \lambda \end{bmatrix} = [\text{localSolverMat}] \begin{bmatrix} \widehat{\mathbf{u}}^{\mathbf{F}_{i1}} \\ \vdots \\ \widehat{\mathbf{u}}^{\mathbf{F}_{in}} \\ \rho_i \end{bmatrix} + [\text{localSolverVec}]$$

  - Computation of elemental matrix and vector

$$\mathbf{K}^i = \begin{bmatrix} [A_{\hat{u}u}\ A_{\hat{u}L}\ A_{\hat{u}p}]*[\text{localSolverMat}] + [A_{\hat{u}\hat{u}}, \mathbf{0}] \\ A_{p\hat{u}} \qquad\qquad 0 \end{bmatrix}$$

$$\mathbf{f}^i = \begin{bmatrix} -[A_{\hat{u}u}\ A_{\hat{u}L}\ A_{\hat{u}p}]*[\text{localSolverVec}] \\ 0 \end{bmatrix}$$

and assembly in the 3 faces (with flipping)

- Modification of the system to impose Dirichlet boundary conditions
- Solution of the system
- Computation of superconvergent velocity
- Postprocess

Remarks:

- L2 projection of the Dirichlet data, otherwise superconvergent solution with order only $h^{k+1.5}$
- The superconvergent solution can be computed as for Laplace or with a formulation that takes into account incompressibility